

WAVELETS AND THE SQUEEZED STATES OF QUANTUM OPTICS*

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ABSTRACT

Wavelets are new mathematical objects which act as “designer trig functions.” To obtain a wavelet, the original function space of finite energy signals is generalized to a phase-space, and the translation operator in the original space has a scale change in the new variable adjoined to the translation. Localization properties in the phase-space can be improved and unconditional bases are obtained for a broad class of function and distribution spaces. Operators in phase space are “almost diagonal” instead of the traditional condition of being diagonal in the original function space. These wavelets are applied to the squeezed states of quantum optics. The scale change required for a quantum wavelet is shown, with Prof. G.M. D’Ariano, to be a Yuen squeeze operator acting on an arbitrary density operator.

1. INTRODUCTION

Wavelets were created in France less than a decade ago¹⁻⁵ when J. Morlet^{1,4} generalized the phase-space of Gabor⁶ by adding a scale change to the frequency (wavenumber) axis for applications to geophysical exploration. Grossmann^{2,5} and Meyer^{3,5,7} immediately saw the importance of wavelets for mathematical physics and to deep questions in harmonic analysis, respectively. There are a number of review articles^{4,5,7-12} available today, each specializing in different aspects of wavelets.

In terms of this paper, which applies wavelets to the squeezed states of quantum optics, two long mathematics papers are the most important. The author is convinced that they will also be the most important for physics, applied mathematics, engineering and industrial problems. The two key papers are those of Daubechies¹³, and Frazier and Jawerth.¹⁴ Daubechies¹³ first constructed a large family of orthonormal bases of compactly supported wavelets in $L^2(R^n)$. Frazier and Jawerth¹⁴ gave a thorough, complete treatment of **sampled** wavelets which is valid both in the classical function spaces and in the modern distributional spaces.

The approach to squeezed states and quantum optics¹⁵⁻²² will be through the coherent states.²³⁻²⁵ The three main approaches to coherent states are those due to Klauder,^{23,26-29} to Perelomov,²⁸ and to Onofrio.²⁹ The Klauder construction starts with an arbitrary representation of a Lie group G on a complex separable Hilbert space \mathcal{H} and induces a representation of G on itself with \mathcal{H} as a *closed* subspace of $L^2(R^d, d\mu)$. This yields a subrepresentation of the regular representation in the sense of Mackey.³⁰ It works equally well with states or frames. The approach of Perelomov starts with a “Little vector” and requires a multiplier to add enough structure to force projective representation to be unitary. There is additional subtlety in obtaining an invariant measure on the coset spaces used to reduce G in that G/H_1 can have an invariant measure $d\mu_1$ whereas G/H_2 may not. Thus, the choice of a “Little group” or “stability subgroup” is a sensitive issue in the Perelomov approach. The Onofrio construction yields a holomorphic representation of the Lie group. At

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least in simple cases it gives a complexification of the real homogeneous space $M = G/H$, of Perelomov. For additional structure and the proofs see the nice new monograph of Kaiser.³¹

2. SQUEEZED STATES OF QUANTUM OPTICS

The coherent states for each complex number α are generated from the unique, translationally invariant Fock vacuum $|0\rangle$ using a unitary displacement operator $D(\alpha)$ which is defined below. Let a, a^+ be the Bose destruction and creation operator which satisfy the canonical commutation relations

$$[a, a^+] = 1 \quad ,$$

$$[a, a] = [a^+, a^+] = 0 \quad , \quad (1)$$

and define D as a Weyl-Heisenberg operator,

$$D(\alpha) := \exp(\alpha a^+ - \alpha^* a) \quad . \quad (2)$$

Then

$$|\alpha\rangle = D(\alpha) |0\rangle \quad (3)$$

is the ordinary coherent state. In terms of an additional complex parameter ζ , the two-photon squeezed states $|\zeta, \alpha\rangle$ of Stoler¹⁵ and Yuen¹⁶ can be generated using the squeezing operator $S(\zeta)$

$$S(\zeta) := \exp(\zeta a^{+2} - \zeta^* a^2) \quad (4)$$

through the action

$$\begin{aligned} |\zeta, \alpha\rangle &= S(\zeta) |\alpha\rangle \\ &= S(\zeta) D(\alpha) |0\rangle \quad . \end{aligned} \quad (5)$$

The states generated in Eq. (5) will be called amplitude squeezed states. These coherent states satisfy the **uncertainty principle** but squeeze one side, say time or frequency, exponentially. *Naively*, it would seem that higher order squeezed operators $S^{(k)}$, $k > 2$, can be defined through the definition

$$S^{(k)}(\zeta) := \exp(\zeta a^{tk} - \zeta^* a^k) \quad , \quad (6)$$

but a neat paper by Fisher, Nieto and Sundberg³² has shown the matrix element divergence

$$\langle 0 | S^{(k)}(\zeta) | 0 \rangle \rightarrow \infty \quad , \quad (7)$$

for all $k > 2$! This can be interpreted as either non-analyticity of the vacuum or as operator domain problems. The task of defining k -photon squeezed wavelets will be relegated to future works. Here a new **quantum** or **operator-valued wavelet** of D'Ariano and the author³³ will be presented. Let \hat{A} be an observable

$$\hat{A} |a\rangle = a |a\rangle \quad (8)$$

where the states $|a\rangle$ give a resolution of the identity

$$1 = \int d\mu(a) |a\rangle \langle a| \quad , \quad (9)$$

where $d\mu(a)$ is the invariant measure. The generating function of moments of the observable \hat{A} in a state whose density operator is $\hat{\rho}$ is given by

$$\langle e^{i\alpha\hat{A}} \rangle := \text{Tr}[e^{i\alpha\hat{A}}\hat{\rho}] \quad . \quad (10)$$

The probability distribution function $P(\rho, a)$ is defined as

$$P(\hat{\rho}, a) := \text{Tr}[|a\rangle\langle a| \hat{\rho}]$$

and is the Fourier transform of the generating function of moments with respect to the measure $d\mu(a)$

$$\langle e^{i\alpha\hat{A}} \rangle = \int d\mu(a) e^{i\alpha a} P(\hat{\rho}, a) \quad . \quad (11)$$

A filtered Fourier transform with window function $\gamma(a)$ for Eq. (11) can be defined naturally as

$$\left\langle e^{i\alpha\hat{A}} \right\rangle_{\gamma} := \int d\mu(\theta) e^{i\alpha a} \gamma(a) P(\hat{\rho}, a) \quad . \quad (12)$$

A c-number wavelet transform analogous to Eq. (12) is given by

$$W(\hat{\rho}, \eta, \epsilon) := \frac{1}{|\epsilon|^{1/2}} \int d\mu(\theta) \chi\left(\frac{a-\eta}{\epsilon}\right) P(\hat{\rho}, a) \quad . \quad (13)$$

In the next section, additional discussions of wavelets will be given.

3. WAVELETS

For simplicity of exposition, let $f \in L^2(R^1)$ be a real or complex-valued finite energy signal and denote its Fourier transform by $\hat{f}(k)$. In L^2 , \hat{f} is guaranteed to exist and by Parseval's theorem $|f|^2 = |\hat{f}|^2$ with proper normalization. Choose conventions s.t.

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{2\pi i k \cdot x} dx \quad , \quad (14)$$

and

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{-2\pi i k \cdot x} dk \quad . \quad (15)$$

If $\text{supp}(\hat{f}) \subset (-1/2, 1/2)$ and $\hat{f} \in L^2(\hat{R}^1)$

$$f(x) = \sum_k \left\{ f(k) \frac{\sin(\pi(x-k))}{\pi(x-k)} \right\} \quad . \quad (16)$$

Inverting Eq. (16) yields

$$\hat{f}(k) = \left\{ \sum_r f(r) e^{2\pi i r k} \right\} \chi_{(-1/2, 1/2)}(r) \quad , \quad (17)$$

where χ_A is the characteristic function of the interval $A \subset R^1$. With discretization the mn^{th} coefficient of f (m, n integers) with “window function” $g(x - nx_0)$, which is one when $(x - nx_0)$ is positive, is given by

$$C_{mn}(f) = \int_{-\infty}^{\infty} e^{2\pi i m k_0 x_0} g(x - nx_0) f(s) dx \quad . \quad (18)$$

Observe that the n -index is a spatial translation of units of x_0 and the m -index is a wave-number translation in units of k_0 . The joint appearance of (m, n) indicates that C_{mn} lives in $\hat{Z}^d \otimes Z^d$, a phase-space. The scale change $x \rightarrow x/2^\nu$ plays an important role in the Calderón complex interpolation approach. A **dilation** is a translation in x (space) with a scale change in Fourier transform variable k . Calderón³⁵ published his famous reproducing formula in 1964. The conditions required are the following: (i) Let ψ and φ be radial, smooth $L^2(R^d)$ functions whose Fourier transforms $\hat{\psi}$ and $\hat{\varphi}$ in $L^2(\hat{R}^2)$ with support in a set A ,

$$A := \text{supp}(\hat{\psi}, \hat{\varphi}) = \left\{ k \mid 0 < C_1 \leq |k| \leq C_2 < \infty \right\} \quad . \quad (19)$$

(ii) For each $|k| \neq 0$

$$\sum_{\nu=-\infty}^{\infty} \hat{\varphi}(2^\nu |k|) \hat{\psi}(2^\nu |k|) = 1 \quad . \quad (20)$$

Let $f \in L^2(R^d)$ with $\hat{f}(0) = 0$ and then

$$\hat{f}(k) = \sum_{\nu} \hat{f}(k) \hat{\varphi}(2^\nu |k|) \hat{\psi}(2^\nu |k|) \quad . \quad (21)$$

Let

$$g_\nu(k) = \hat{f}(k) \hat{\varphi}(2^\nu |k|) \quad (22)$$

which implies that rearrangement of Eq. (21) into

$$\sum_s c_s(k) e^{2\pi i 2^\nu k} \quad (23)$$

has an obvious parallel to Eqs. (14-18). Define the quantities

$$\varphi_{\nu,r}(x) := 2^{-\nu/2} \varphi(2^{-\nu} x - r) \quad , \quad (24)$$

$$\psi_{\nu,r}(x) := 2^{-\nu/2} \psi(2^{-\nu} x - r) \quad , \quad (25)$$

$$\varphi_\nu(x) := 2^{-\nu/2} \varphi\left(\frac{x}{2^\nu}\right) \quad (26)$$

and use these with Eq. (23) to obtain

$$c_s(k) = \varphi_k * f(s \cdot 2^k) := \langle f, \varphi_{k,s} \rangle \quad . \quad (27)$$

Now

$$f(x) = \sum_k \sum_s \langle f, \varphi_{k,s} \rangle \psi_{k,s}(x) \quad , \quad (28)$$

is a continuous wavelet expansion with wavelet coefficients given by the $c_s(k)$'s of Eq. (27). The mathematical importance of the wavelet expansion over the Fourier method, is that it generalizes to many function and distribution spaces where Fourier analysis is inapplicable. The potential physical importance of wavelet methods is to make possible new formulations and calculations of physical models. For computation or for experimental signal processing the discrete wavelet transform of Frazier-Jawerth called the φ -**transformation** is required. The φ -transform of f is

$$(f, \varphi, \psi) \longrightarrow \sum_k \sum_s \langle f, \varphi_{k,s} \rangle \psi_{k,s} \quad (29)$$

and holds in general function and distribution spaces. The requirements on the L^2 function which φ must satisfy are:

$$(i) \quad \int_{-\infty}^{\infty} \varphi(x) dx = 0 \quad , \quad (30)$$

$$(ii) \quad \varphi(x) = \varphi_{r,s}(x) = \varphi\left(\frac{x-s}{r}\right) \quad , \quad (31)$$

$$(iii) \quad c_{r,s}(\varphi) = 2\pi \int_{-\infty}^{\infty} |\varphi(k)|^2 \frac{dk}{|k|} < \infty \quad . \quad (32)$$

The *inverse problem* of reconstructing $f(x)$ from coefficients^{1,2,4,5,12} can then be reduced to the matrix problem

$$f(x_i) = \left[c_{r,s}(\varphi) \right]^{-1} \left[c_{s,j}(f) \right] \cdot \bar{\psi}\left(\frac{x_i - j}{s}\right) \quad . \quad (33)$$

Observe the correspondence of 2^ν with a scale change and $r(r/s)$ with a spatial translation. A major improvement of wavelets over Fourier methods is apparent from Eq. (16). Whereas, $\hat{f}(k)$ has great *localization* with compact support, $f(x)$ has terrible localization properties in x since as $x \rightarrow \infty$

$$\frac{\sin(\pi x)}{\pi x} \longrightarrow \frac{1}{|x|} \quad . \quad (34)$$

The Frazier-Jawerth improvement in localization via the φ -transform is easily seen for $\hat{\psi}(k) \in C_0^r(R)$, where the integration by parts

$$(2\pi i x) \cdot f(x) = - \int_{-\infty}^{\infty} \left(\frac{\partial f}{\partial k} \right)(k) e^{-2\pi i k \cdot x} dk \quad , \quad (35)$$

can be repeated r -times to obtain

$$(2\pi i x)^r f(x) = (-1)^r \int_{-\infty}^{\infty} \left(\frac{\partial^r f}{\partial k^r} \right)(k) e^{-2\pi i k \cdot x} dk \quad . \quad (36)$$

Hence, as $x \rightarrow \infty$

$$|f(x)| = O\left(\frac{1}{|x|^r}\right) . \quad (37)$$

The size of $c_{r,s}(f, \varphi)$ depends largely on f in a neighborhood of the point $(0, \frac{s}{2^r})$ with a spread (2^{-r}) and far from this point the coefficient decays as $|x|^{-s}$. The simplest way to obtain wavelets is to decompose the space of interest V_1 into a closed subspace V_0 and its orthogonal complement W_0 according to the direct sum

$$V_1 = V_0 \oplus W_0 , \quad (38)$$

or schematically

$$\begin{array}{c} V_1 \\ \swarrow \quad \searrow \\ V_0 \quad W_0 \end{array}$$

In order to maintain simplicity let V_1 denote either L^2 or ℓ^2 . Let φ be a given function in V_1 which satisfies the relation

$$\varphi(x) = \sum_n h(n) 2^{1/2} \varphi(2x - n) , \quad (39)$$

where the set of coefficients $\{h(n)\}_{n=1}^N$ are a collection of constants, called “masking coefficients” and the $2^{1/2}$ factor in front of φ is for L^2 normalization. If V_1 is the closed, linear span of all functions $\{2^{1/2} \cdot \varphi(2x - n)\}$,

$$V_1 = \overline{V_n \{2^{1/2} \varphi(2 \cdot - n)\}} \quad (40)$$

and $V_1 \simeq \ell_2$ and is a (useful) special case of Frazier-Jawerth.

Proposition: If the masking coefficients satisfy the condition

$$\sup_n \left\{ \sum_k |h(k - 2n)|^2 \right\} \leq A \quad (41)$$

for $0 < A < \infty$ and $A \in R^1$ then the space V_0 of Eq. (39) is given by

$$V_0 = V_n \varphi(\cdot - n) \quad (42)$$

with $V_0 \simeq \ell_2$ and $V_0 \subset V_1$.

Proof: Any $f(x) \in V_1$ can be expanded in $\varphi(\cdot - n)$'s

$$\begin{aligned} f(x) &= \sum_n c_n \varphi(x - n) \\ &= 2^{1/2} \sum_n \sum_m c_n h(m) \varphi(2(x - n) - m) \\ &= 2^{1/2} \sum_n \sum_m c_n h(m) \varphi(2x - m - 2n) . \end{aligned} \quad (43)$$

Let

$$\begin{aligned} k &= m + 2n \\ f(x) &= 2^{1/2} \sum_k b_k \varphi(2x - k) \end{aligned} \quad (44)$$

where

$$b_k = \sum_n c_n h(k - 2n) \quad , \quad (45)$$

and take

$$\begin{aligned} \sum_k |b_k|^2 &= \sum_k \left[\sum_n c_n h(k - 2n) \right]^2 \\ &\leq \sum_k \cdot \sum_n |c_n|^2 |h(k - 2n)|^2 \quad . \end{aligned} \quad (46)$$

The requirement in Eq. (42) suffices for

$$\sum_k |b_k|^2 \leq A \left(\sum_n |c_n|^2 \right) \quad , \quad (47)$$

for every $\varphi \in V_1$. In the event that $\{h(n)\}_{n=1}^\infty \in \ell^2$, eq. (42) is automatically satisfied and V_0 is a closed subspace of V_1 .

Question: Does a function $\psi \in V_0^\perp$ exist s.t.

- (i) $W_0 = V_0^\perp = \overline{V\psi(\cdot - n)}$ and
- (ii) $\{\psi(\cdot - n)\}_{n=1}^\infty$ is an orthonormal basis.

Answer: Yes; Daubechies¹³ in L^2 , by Frazier and Jawerth¹⁴ in ℓ^2 , Besov spaces, Sobolev spaces, bounded mean oscillation (BMO) spaces and Triebel-Lizorkin spaces. Such a function ψ is a **wavelet** and

$$\{2^{\nu/2} \psi(2^\nu \cdot - n) : \nu, n \in \mathbb{Z}\} \quad , \quad (48)$$

is an orthonormal basis for $L^2(\mathbb{R})$. This reduces the problem to that of finding a finite set of masking coefficients $\{h(n)\}_{n=1}^N$. The easiest method for finding these coefficients is due to Daubechies in Ref. (13). Assume that the set of non-zero masking coefficients is a finite set and let $\varphi \in L^2$ s.t.

$$\hat{\varphi}(0) = 1 \quad . \quad (49)$$

Given the wavelet expansion

$$\varphi(x) = \sum h(n) 2^{1/2} \varphi(2x - n) \quad ,$$

take its Fourier transform to obtain

$$\begin{aligned} \hat{\varphi}(k) &= \sum h(n) e^{2\pi i n k/2} \varphi(k/2) / 2^{1/2} \\ &=: m(k/2) \varphi(k/2) \quad . \end{aligned} \quad (50)$$

It is now necessary to show that

$$\begin{aligned} \hat{\varphi}(k) &= \lim_{N \rightarrow \infty} \left\{ \prod_{j=1}^N m(k/2^j) \cdot \hat{\varphi}(k/2^N) \right\} \\ &= \prod_{j=1}^\infty m(k/2^j) \end{aligned} \quad (51)$$

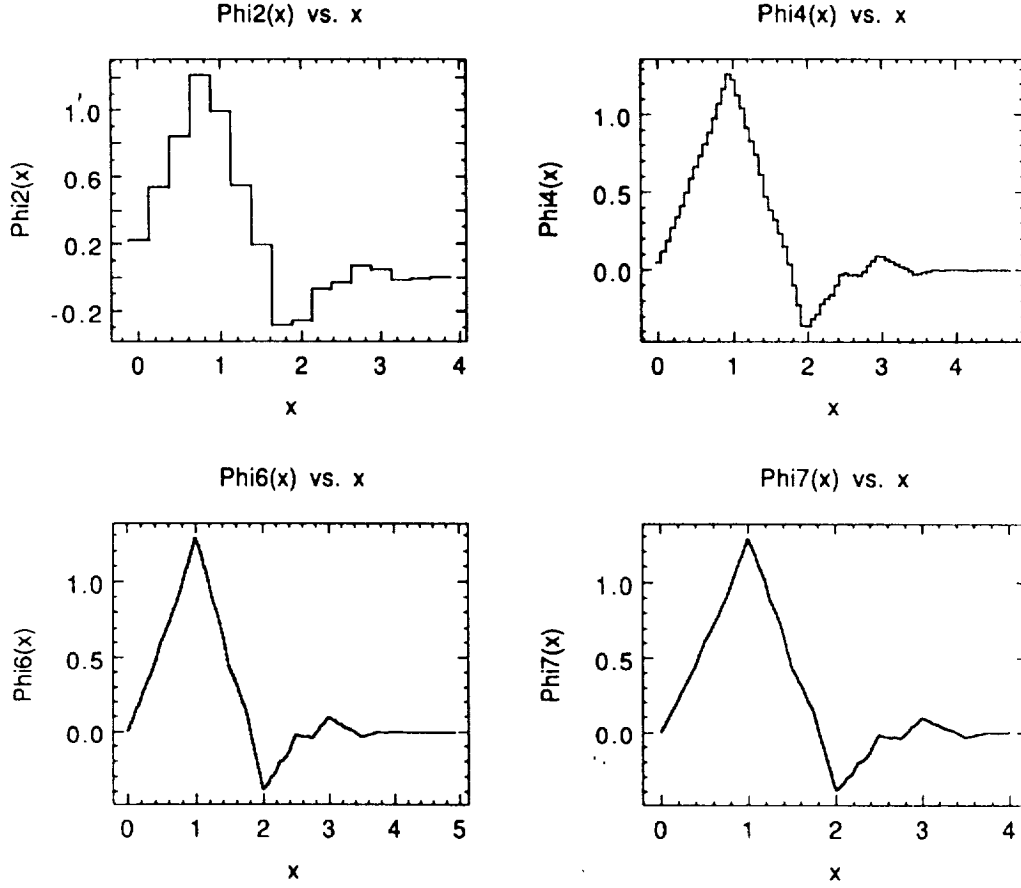


Fig. 1

The second, fourth, sixth and seventh iteration of the Box function for $\varphi(x)$.

makes sense in L^2 (or V_1). This suggests a method of finite approximation providing the masking coefficients are known. Let $\eta_0(x) \in L^2(V_1)$ s.t.

$$\hat{\eta}_0(0) = \int_{-\infty}^{\infty} \eta_0(x) dx = 1 \quad , \quad (52)$$

and iterate

$$\eta_r(x) = \sum_n h(n) 2^{1/2} \varphi_{r-1}(2x - n) \quad , \quad (53)$$

to generate φ which is a wavelet, but is not the wavelet ψ of Eq. (25), but rather is that of φ in Eq. (24) instead. In Daubechies nomenclature φ is called a *father wavelet* and if x is identified as a “time” variable the dilations (= scale changes and translations) of φ span V_0 which acts as the high frequency, $k = \omega$, content of the full space V_1 . The function φ can be thought of as a “pixel shape” in V_0 as pointed out by Kaiser.³¹ Similarly, ψ is called the *mother wavelet* and the dilations of ψ span W_0 , which contains the low frequency content of V_1 . In Figs. 1, and 2 a mother and father wavelet generated by the choice $\eta_0(x) = b(x)$, the Box function

$$b(x) = \begin{cases} 1, & 1/2 < x < 1/2 \\ 0, & \text{otherwise} \end{cases} \quad . \quad (54)$$

Parent Wavelets

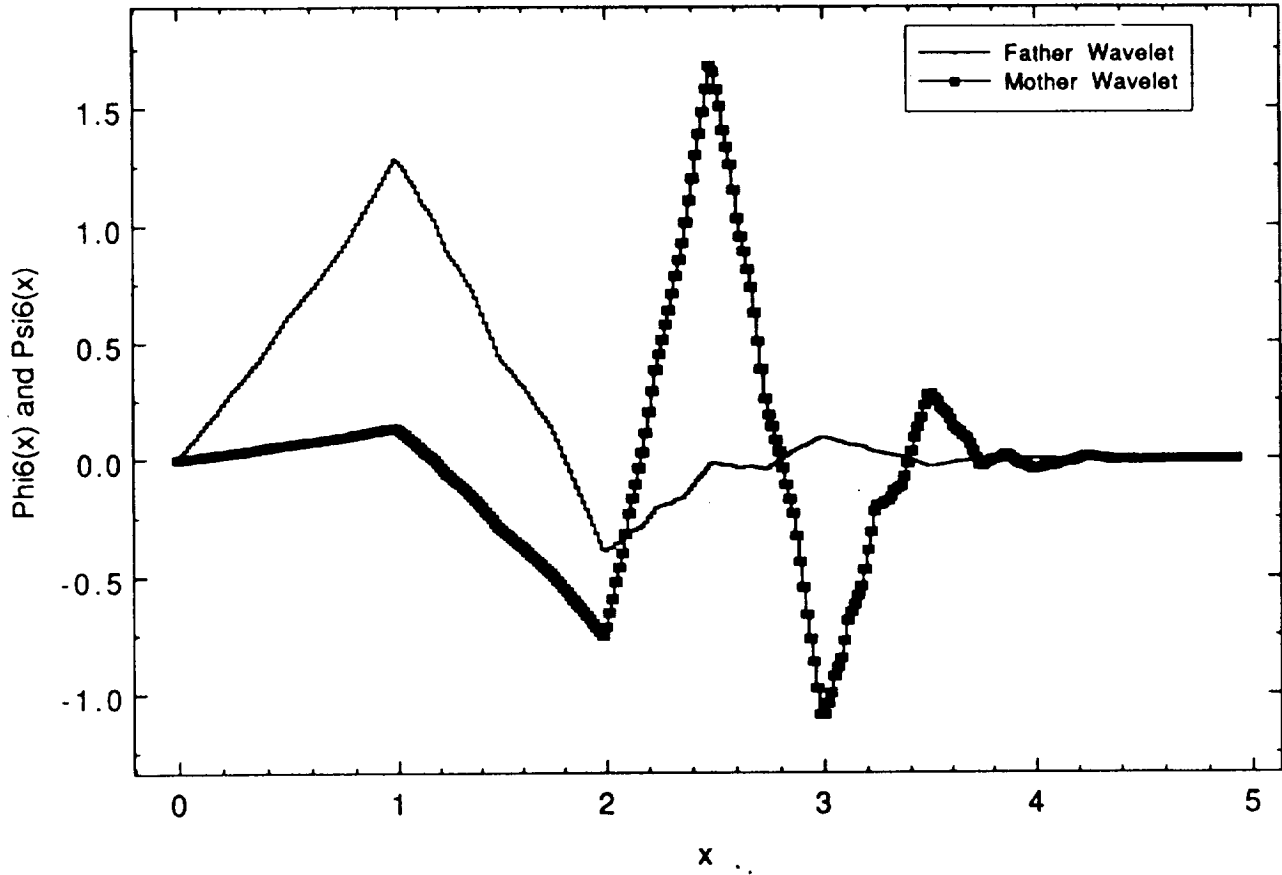


Fig. 2

The father wavelet $\varphi(x)$ and the mother wavelet $\psi(x)$ for η_0 equal to the Box function.

Since $\psi \in W_0 \subset V_1$,

$$\begin{aligned}\psi(x) &= \sum \tilde{h}(n) 2^{1/2} \varphi(2x - n) \\ \hat{\psi}(k) &= \tilde{m}(k/2) \cdot \hat{\varphi}(k)\end{aligned}\tag{55}$$

where $\{\tilde{h}(n)\}_{n=1}^N$ is a set of masking coefficients for W_0 and $\tilde{m}(\cdot)$ is a function analogous to $m(\cdot)$ in Eq. (47). Let $\tau = \pi$ be a translation parameter and observe that finding φ and ψ is equivalent to finding two trigonometric polynomials s.t. the 2×2 matrix

$$u(k) = \begin{pmatrix} m(k) & \tilde{m}(k) \\ m(k + \tau) & \tilde{m}(k + \tau) \end{pmatrix}\tag{56}$$

is unitary. It is useful to consider $m(k)$ as a phase function which partitions by translations,

$$|m(k)|^2 + |m(k + \tau)|^2 = 1 \quad .\tag{57}$$

To solve for $|m(k)|^2 = P(|\sin x|^2, |\cos x|^2)$ treat P as the probability of a binomial process with possible outcomes

$$\begin{aligned}p_1 &= |\sin x|^2 \\ p_2 &= |\cos x|^2 \quad .\end{aligned}\tag{58}$$

Then $m(k)$ is the square root of P . There are many solutions since P is an even, positive polynomial but only one is needed. Then using m , it is straightforward to find \tilde{m} . This completes the discussion of simple wavelets.

For mathematical physics, operators and their expectations are the objects of interest. The spectra of operators give infinite dimensional "diagonalizations" in terms of generalized eigenfunctions. In a wavelet basis an operator is "almost diagonal" in a sense discussed next.

Let T be an operator, f a function in a normed space which is in the dense domain of T ,

$$\begin{aligned} T : f &\rightarrow Tf \\ (Tf)(x) &= \int K(x, y) f(y) dy \end{aligned} \quad (59)$$

s.t.

$$(i) \quad |K(x, y)| \leq \frac{c}{|x - y|} \quad , \quad (60a)$$

and

$$(ii) \quad \left| \frac{\partial K(x, y)}{\partial x} \right| + \left| \frac{\partial K(x, y)}{\partial y} \right| \leq \frac{c}{|x - y|^2} \quad . \quad (60b)$$

Let

$$f(x) = \sum_{j,k} c_{jk} \psi_{jk}(x) \quad (61)$$

where the $\psi_{jk}(x)$'s are a wavelet basis. Then the kernel of T can be written as

$$K(x, y) = \sum_{j,k} e_{jk} \psi_{jk}(x) \bar{\psi}_{jk}(y) \quad (62)$$

where the ψ_{jk} 's satisfy the estimate

$$|\psi_{jk}(x)| < \frac{1}{2^{j/2}} (1 + |2^j \cdot x - k|)^{-1-\epsilon} \quad . \quad (63)$$

In order to prove that K satisfies the conditions (i) and (ii) split the sum according to

$$\sum_{j,k} = \sum_{j < j_0, k} + \sum_{j \geq j_0, k} \quad (64)$$

with j_0 chosen so that

$$2^{j_0} \leq |x - y| \leq 2^{j_0+1} \quad . \quad (65)$$

Using the decomposition of Eq. (64), the estimate of Eq. (63) and geometrical sums, it follows that

$$|K(x, y)| \leq \frac{c}{|x - y|} \quad . \quad (66)$$

The same techniques yield

$$\left| \frac{\partial K(x, y)}{\partial x} \right| \leq \frac{c}{|x - y|^2} \quad , \quad (67)$$

and

$$\left| \frac{\partial K(x, y)}{\partial y} \right| \leq \frac{c}{|x - y|^2} \quad . \quad (68)$$

There are several consequences of conditions (i), (ii) and Eq. (63). One is that such operators map $L^p \rightarrow L^p$ for all $1 < p < \infty$, solving deep, old problems. Another is that $\{\psi_{jk}\}$ are an unconditional basis for $L^p, 1 < p < \infty$. The proofs work because of phase-space localization; if two frequencies are well separated their wavelet coefficients are small and if two times are well separated their wavelet coefficients are small. The localization structure in Eq. (64) is the reason that the disadvantages of “almost diagonal” are outweighed by the advantages.

4. WAVELETS FOR SQUEEZED STATES

It is clear that in order to define a wavelet for the squeezed states of quantum optics, it is necessary to define an operator which changes the scale. This has been accomplished in Ref. (33) in a project with G.M. D’Ariano which was initiated at this workshop.

Let $\chi(\cdot)$ be an analytic function of the observable \hat{A} defined s.t.

$$\chi(\hat{A})|a\rangle = a|a\rangle \quad (69)$$

and

$$\chi(\hat{A}) := \int d\mu(a) \chi(a) |a\rangle\langle a| \quad . \quad (70)$$

This function satisfies the relation

$$\begin{aligned} \langle \chi(\hat{A}) \rangle &= \text{Tr}[\chi(\hat{A})\hat{\rho}] \\ &= \int d\mu(a) \chi(a) P(\hat{\rho}, a) \quad . \end{aligned} \quad (71)$$

A dilation operator in the Heisenberg picture is defined on an observable \hat{A} as

$$\mathcal{D}_{\eta\epsilon}(\hat{A}) := \frac{\hat{A} - \eta}{\epsilon} \quad . \quad (72)$$

The time picture is suppressed since no other picture will be used here although Schrödinger picture operators are given in Ref. (17). For some observables, \hat{A} , the dilation operator is **unitary** but D’Ariano^{21,34} has shown that there are important operators of quantum optics that are **completely positive** maps, abbreviated CP, and are non-unitary. In the unitary case, which is the only case discussed here, all products of operators are preserved and

$$\begin{aligned} \mathcal{D}_{\eta\epsilon}(\hat{A}) &= \chi\left\{\mathcal{D}_{\eta\epsilon}(\hat{A})\right\} \\ &= \chi\left(\frac{\hat{A} - \eta}{\epsilon}\right) \quad . \end{aligned} \quad (73)$$

Using Eq. (73), the operator-valued wavelet transform can now be written as

$$\begin{aligned} W(\hat{\rho}; \eta, \epsilon) &:= \frac{1}{|\epsilon|^{1/2}} \text{Tr} \{ \mathcal{D}_{\eta\epsilon}(\hat{A}) \hat{\rho} \} \\ &= \frac{1}{|\epsilon|^{1/2}} \left\langle \mathcal{D}_{\eta\epsilon}(\chi(\hat{A})) \right\rangle . \end{aligned} \quad (74)$$

Thus, the dilation operator **squeezes** any state described by a density operator $\hat{\rho}$. In Ref. (34) two examples are presented:

- (i) The unitary dilation of one quadrature of the electric field. This case is applicable to a phase sensitive amplification.
- (ii) The CP dilation map of the particle number which is applicable to improving noise sensitivity in squeezed light signals.

and analogies of these have been obtained for a quantum wavelet in Ref. (33) with Prof. G.M. D'Ariano.

5. CONCLUSIONS AND OUTLOOK

The scale change part of the wavelet dilation is accomplished by the Yuen¹⁶ squeeze operator. The application of wavelets to quantum optics is an idea with some potential. For example, nonlinear modes and mode-coupling using wavelets should prove useful. The quantum squeezed wavelet with D'Ariano should be a good candidate for highly dispersive biological media. Future work will focus on these ideas.

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